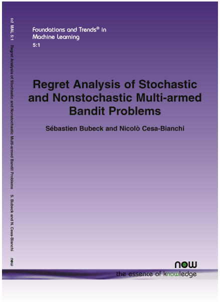
Regret Analysis of Stochastic and Nonstochastic Multi-armed Bandit Problems, Part 2

Sébastien Bubeck

Theory Group

Microsoft Research



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Sébastien Bubeck and Nicolò Cesa-Bianchi

now
the essence of knowledge

The linear bandit problem, Auer [2002]

Known parameters: compact action set $\mathcal{A} \subset \mathbb{R}^n$, adversary's action set $\mathcal{L} \subset \mathbb{R}^n$, number of rounds T.

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Other models: In the i.i.d. model we assume that there is some underlying $\theta \in \mathcal{L}$ such that $\mathbb{E}(Y_t | a_t) = \theta^\top a_t$. In the Bayesian model we assume that we have a prior distribution ν over the sequence $(\ell_1, \dots, \ell_T)$ (in this case the expectation in R_T is also over $(\ell_1, \dots, \ell_T) \sim \nu$). Alternatively we could assume a prior over θ.

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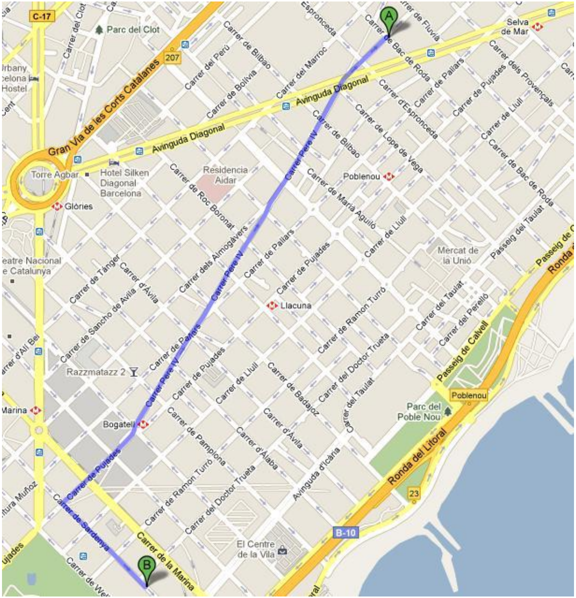
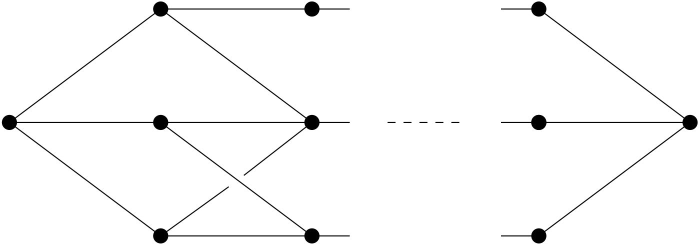
Assumption: unless specified otherwise we assume $\mathcal{L} = \mathcal{A}^\circ := \{\ell : \sup_{a \in \mathcal{A}} |\ell^\top a| \leq 1\}$.

Example: path planning



Example: path planning

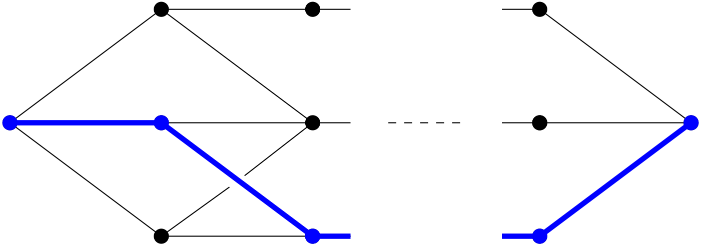
Adversary



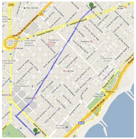
Player

Example: path planning

Adversary

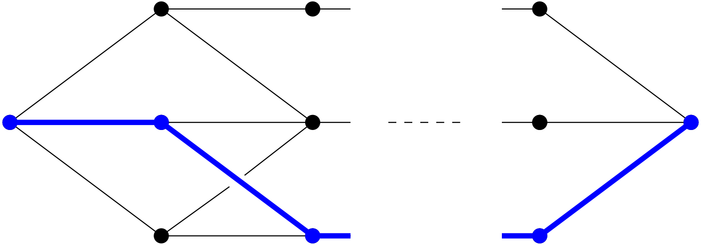


Player →



Example: path planning

Adversary ⟶



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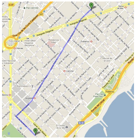

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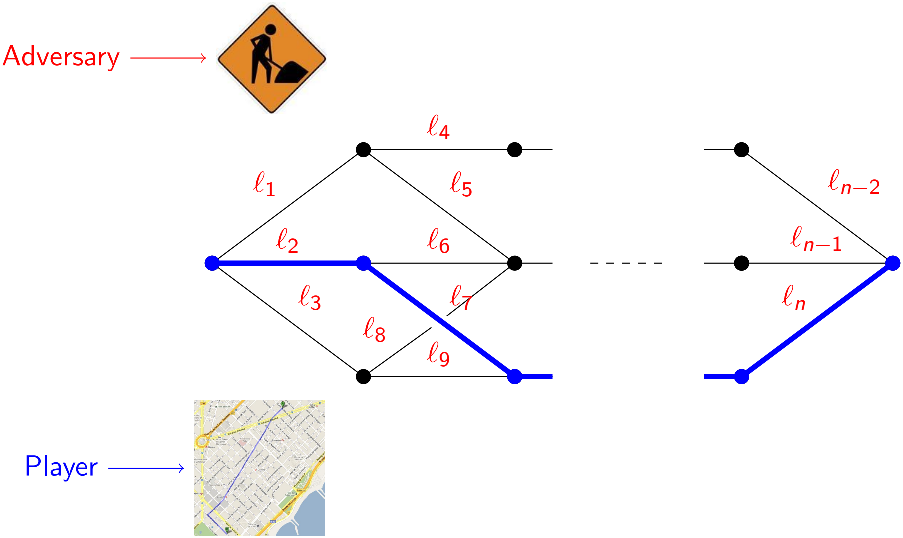

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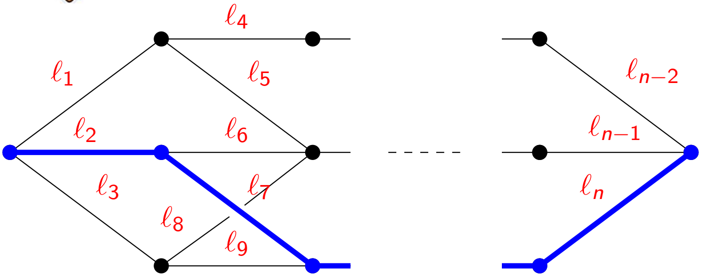

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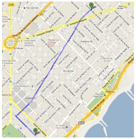

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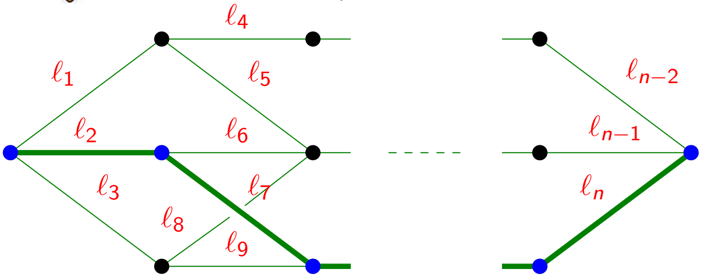

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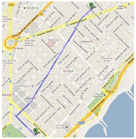

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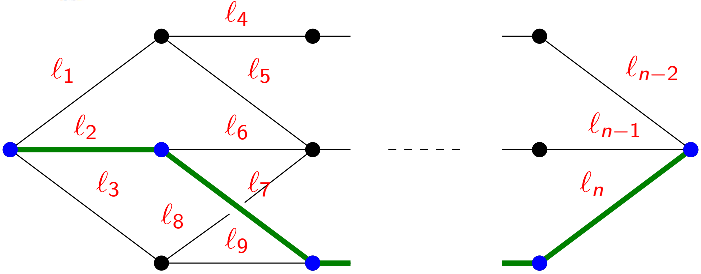
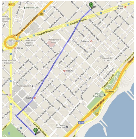


Player ⟶ 

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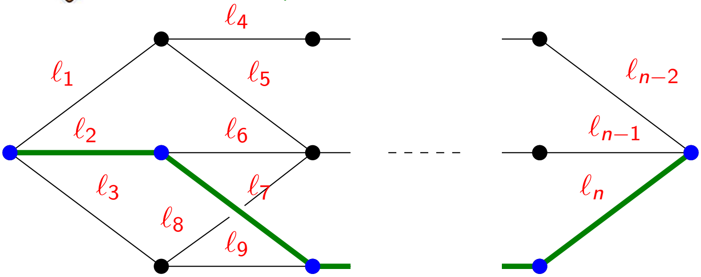

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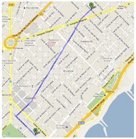

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Thompson Sampling for linear bandit after RVR14

Assume $\mathcal{A} = \{a_1, \dots, a_{|\mathcal{A}|}\}$. Recall from Part 1 that TS satisfies

$$\sum_i \pi_t(i)(\bar{\ell}_t(i) - \bar{\ell}_t(i,i)) \leq \sqrt{C \sum_{i,j} \pi_t(i)\pi_t(j)(\bar{\ell}_t(i,j) - \bar{\ell}_t(i))^2}$$

$$\Rightarrow R_T \leq \sqrt{C \ T \ \log(|\mathcal{A}|)/2},$$

where $\bar{\ell}_t(i) = \mathbb{E}_t \ell_t(i)$ and $\bar{\ell}_t(i,j) = \mathbb{E}_t(\ell_t(i)|i^* = j)$.

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Writing $\bar{\ell}_t(i) = a_i^\top \bar{\ell}_t$, $\bar{\ell}_t(i,j) = a_i^\top \bar{\ell}_t^j$, and $(M_{i,j}) = \left(\sqrt{\pi_t(i)\pi_t(j)} a_i^\top (\bar{\ell}_t - \bar{\ell}_t^j)\right)$ we want to show that

$$\mathrm{Tr}(M) \leq \sqrt{C} \|M\|_F .$$

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Using the eigenvalue formula for the trace and the Frobenius norm one can see that $\mathrm{Tr}(M)^2 \leq \mathrm{rank}(M) \|M\|_F^2$. Moreover the rank of M is at most n since $M = UV^\top$ where $U, V \in \mathbb{R}^{|\mathcal{A}| \times n}$ (the i^{th} row of U is $\sqrt{\pi_t(i)} a_i$ and for V it is $\sqrt{\pi_t(i)}(\bar{\ell}_t - \bar{\ell}_t^i)$).

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1. TS satisfies $R_T \leq \sqrt{nT \log(|\mathcal{A}|)}$. To appreciate the improvement recall that without the linear structure one would get a regret of order $\sqrt{|\mathcal{A}| T}$ and that $\mathcal{A}$ can be exponential in the dimension n (think of the path planning example).

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2. Provided that one can efficiently sample from the posterior on ℓ_t (or on θ), TS just requires at each step one linear optimization over $\mathcal{A}$.
3. TS regret bound is optimal in the following sense. W.l.og. one can assume $|\mathcal{A}| \leq (10T)^n$ and thus TS satisfies $R_T = O(n\sqrt{T\log(T)})$ for any action set. Furthermore one can show that there exists an action set and a prior such that for any strategy one has $R_T = \Omega(n\sqrt{T})$, see Dani, Hayes and Kakade [2008], Rusmevichientong and Tsitsiklis [2010], and Audibert, Bubeck and Lugosi [2011, 2014].

Adversarial linear bandit after Dani, Hayes, Kakade [2008]

Recall from Part 1 that exponential weights satisfies for any $\widetilde{\ell}_t$ such that $\mathbb{E}\widetilde{\ell}_t(i) = \ell_t(i)$ and $\widetilde{\ell}_t(i) \geq 0$,

$$R_T \leq \frac{\max_i \mathrm{Ent}(\delta_i \| p_1)}{\eta} + \frac{\eta}{2} \mathbb{E} \sum_t \mathbb{E}_{I \sim p_t} \widetilde{\ell}_t(I)^2 .$$

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Up to the issue that $\widetilde{\ell}_t$ can take negative values this suggests the "optimal" $\sqrt{nT\log(|\mathcal{A}|)}$ regret bound.

Adversarial linear bandit, further development

1. The non-negativity issue of $\widetilde{\ell}_t$ is a manifestation of the need for an added exploration. DHK08 used a suboptimal exploration which led to an additional $\sqrt{n}$ in the regret. This was later improved in Bubeck, Cesa-Bianchi, and Kakade [2012] with an exploration based on the John's ellipsoid (smallest ellipsoid containing $\mathcal{A}$).

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4. Bubeck and Eldan [2014]'s entropic barrier allows for a much more information-efficient sampling than AHR08. This gives another strategy with optimal regret which is efficient when $\mathcal{A}$ is convex (and one can do linear optimization on $\mathcal{A}$).

Adversarial combinatorial bandit after Audibert, Bubeck and Lugosi [2011, 2014]

Combinatorial setting: $\mathcal{A} \subset \{0,1\}^n$, $\max_a \|a\|_1 = m$, $\mathcal{L} = [0,1]^n$.

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3. ABL11: Exponential weights is provably suboptimal in this setting! This is in sharp contrast with the case where $\mathcal{L} = \mathcal{A}^\circ$.
4. Optimal regret in the semi-bandit case is $\sqrt{mnT}$ and it can be achieved with mirror descent and the natural unbiased estimator for the semi-bandit situation.

Adversarial combinatorial bandit after Audibert, Bubeck and Lugosi [2011, 2014]

Combinatorial setting: $\mathcal{A} \subset \{0,1\}^n$, $\max_a \|a\|_1 = m$, $\mathcal{L} = [0,1]^n$.

1. Full information case goes back to the end of the 90's (Warmuth and co-authors), semi-bandit and bandit were introduced in Audibert, Bubeck and Lugosi [2011] (following several papers that studied specific sets $\mathcal{A}$).
2. This is a natural setting to study FPL-type (Follow the Perturbed Leader) strategies, see e.g. Kalai and Vempala [2004] and more recently Devroye, Lugosi and Neu [2013].
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5. For the bandit case the bound for exponential weights from the previous slides gives $m\sqrt{mnT}$. However the lower bound from ABL14 is $m\sqrt{nT}$, which is conjectured to be tight.

Preliminaries for the i.i.d. case: a primer on least squares

Assume $Y_t = \theta^\top a_t + \xi_t$ where (ξ_t) is an i.i.d. sequence of centered and sub-Gaussian real-valued random variables. The (regularized) least squares estimator for θ based on $\mathbb{Y}_t = (Y_1, \dots, Y_{t-1})^\top$ is, with $\mathbb{A}_t = (a_1 \dots a_{t-1}) \in \mathbb{R}^{n \times t-1}$ and $\Sigma_t = \lambda \mathrm{I}_n + \sum_{s=1}^{t-1} a_s a_s^\top$:

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A basic martingale argument (see e.g., Abbasi-Yadkori, Pál and Szepesvári [2011]) shows that w.p. $\geq 1 - \delta$, $\forall t \geq 1$,

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Note that $\mathrm{logdet}(\Sigma_t) \leq n \log(\mathrm{Tr}(\Sigma_t)/n) \leq n \log(\lambda + t/n)$ (w.l.o.g. we assumed $\|a_t\| \leq 1$).

i.i.d. linear bandit after DHK08, RT10, AYPS11

Let $\beta = 2\sqrt{n \log(T)}$, and $\mathcal{E}_t = \{\theta' : \|\theta' - \hat{\theta}_t\|_{\Sigma_t} \leq \beta\}$. We showed that w.p. $\geq 1 - 1/T^2$ one has $\theta \in \mathcal{E}_t$ for all $t \in [T]$.

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The appropriate generalization of UCB is to select: $(\widetilde{\theta}_t, a_t) = \mathrm{argmin}_{(\theta', a) \in \mathcal{E}_t \times \mathcal{A}} \theta'^{\top} a$ (this optimization is NP-hard in general, more on that next slide).

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To control the sum of squares we observe that:

$$\det(\Sigma_{t+1}) = \det(\Sigma_t) \det(\mathrm{I}_n + \Sigma_t^{-1/2} a_t (\Sigma_t^{-1/2} a_t)^{\top}) = \det(\Sigma_t)(1 + \|a_t\|^2_{\Sigma_t^{-1}})$$

so that (assuming $\lambda \geq 1$)

$$\log\det(\Sigma_{T+1}) - \log\det(\Sigma_1) = \sum_t \log(1 + \|a_t\|^2_{\Sigma_t^{-1}}) \geq \frac{1}{2} \sum_t \|a_t\|^2_{\Sigma_t^{-1}}.$$

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Putting things together we see that the regret is $O(n \log(T) \sqrt{T})$.

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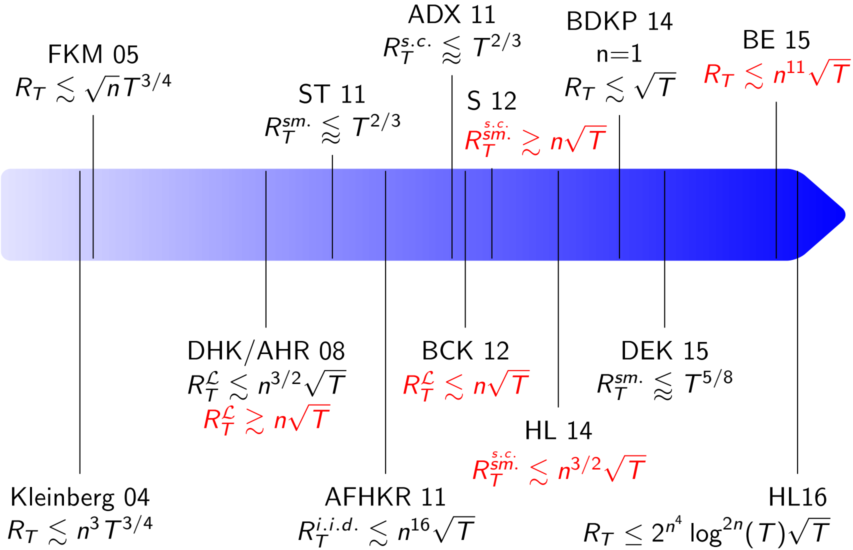
4. $\log(T)$-regime: if $\mathcal{A}$ is finite (note that a polytope is effectively finite for us) one can get $n^2 \log^2(T)/\Delta$ regret:
$$R_T \leq \mathbb{E} \sum_{t=1}^{T} \frac{(\theta^\top(a_t - a^*))^2}{\Delta} \leq \frac{\beta^2}{\Delta} \mathbb{E} \sum_{t=1}^{T} \|a_t\|^2_{\Sigma_t^{-1}} \lesssim \frac{n^2 \log^2(T)}{\Delta}.$$

Some non-linear bandit problems

Lipschitz bandit: Kleinberg, Slivkins and Upfal [2008, 2016], Bubeck, Munos, Stoltz and Szepesvari [2008, 2011];

Gaussian process bandit: Srinivas, Krause, Kakade and Seeger [2010]; and **convex bandit:**



FKM 05
$R_T \lesssim \sqrt{n}T^{3/4}$

ST 11
$R_T^{sm.} \lessapprox T^{2/3}$

ADX 11
$R_T^{s.c.} \lessapprox T^{2/3}$

S 12
$R_T^{\overset{s.c.}{sm.}} \gtrsim n\sqrt{T}$

BDKP 14
n=1
$R_T \lesssim \sqrt{T}$

BE 15
$R_T \lesssim n^{11}\sqrt{T}$

Kleinberg 04
$R_T \lesssim n^3 T^{3/4}$

DHK/AHR 08
$R_T^{\mathcal{L}} \lesssim n^{3/2}\sqrt{T}$
$R_T^{\mathcal{L}} \gtrsim n\sqrt{T}$

AFHKR 11
$R_T^{i.i.d.} \lesssim n^{16}\sqrt{T}$

BCK 12
$R_T^{\mathcal{L}} \lesssim n\sqrt{T}$

HL 14
$R_T^{\overset{s.c.}{sm.}} \lesssim n^{3/2}\sqrt{T}$

DEK 15
$R_T^{sm.} \lessapprox T^{5/8}$

HL16
$R_T \leq 2^{n^4}\log^{2n}(T)\sqrt{T}$

Contextual bandit

We now make the game-changing assumption that at the beginning of each round t a *context* $x_t \in \mathcal{X}$ is revealed to the player. The ideal notion of regret is now:

$$R_T^{\mathrm{ctx}} = \sum_{t=1}^{T} \ell_t(a_t) - \inf_{\Phi:\mathcal{X}\to\mathcal{A}} \sum_{t=1}^{T} \ell_t(\Phi(x_t)).$$

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As far as I can tell the contextual bandit problem is an infinite playground and there is no canonical solution (or at least not yet!). Thankfully all we have learned so far can give useful guidance in this challenging problem.

Linear model after embedding

A natural assumption in several application domains is to suppose linearity in the loss after a correct embedding. Say we know mappings $(\varphi_a)_{a\in\mathcal{A}}$ such that $\mathbb{E}_t(\ell_t(a)) = \varphi_a(x_t)^\top\theta$ for some unknown $\theta \in \mathbb{R}^n$ (or in the adversarial case that $\ell_t(a) = \ell_t^\top \varphi_a(x_t)$).

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This is nothing but a linear bandit problem where the action set is changing over time. All the strategies we described are robust to this modification and thus in this case one can get a regret of $\sqrt{nT \log(|\mathcal{A}|)} \lesssim n\sqrt{T \log(T)}$ (and for the stochastic case one can get efficiently $n^{3/2}\sqrt{T}$).

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A much more challenging case is when the correct embedding $\varphi = (\varphi_a)_{a \in \mathcal{A}}$ is only known to belong to some class Φ. Without further assumptions on Φ we are basically back to the general model. Also note that a natural impulse is to run "bandits on top of bandits", that is first select some $\varphi_t \in \Phi$ and then select a_t based on the assumption that φ_t is correct. We won't get into this here, but let us investigate a related idea.

Exp4, Auer, Cesa-Bianchi, Freund and Schapire [2001]

One can play exponential weights on the set of policies with the following unbiased estimator (obvious notation: $\ell_t(\pi) = \ell_t(\pi(x_t))$, $\pi_t \sim p_t$, and $a_t = \pi_t(x_t)$)

$$\widetilde{\ell}_t(\pi) = \frac{\mathbb{1}\{\pi(x_t) = a_t\}}{\sum_{\pi' : \pi'(x_t) = a_t} p_t(\pi')} \ell_t(a_t).$$

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The only issue of this strategy is that the computationally complexity is linear in the policy space, which might be huge. A year and half ago a major paper by Agarwal, Hsu, Kale, Langford, Li and Schapire was posted, with a strategy obtaining the same regret as Exp4 (in the i.i.d. model) but which is also computationally efficient with an oracle for the offline problem (i.e., $\min_{\pi \in \Pi} \sum_{t=1}^{T} \ell_t(\pi(x_t))$). Unfortunately the algorithm is not simple enough yet to be included in these slides.

The statistician perspective, after Goldenshluger and Zeevi [2009, 2011], Perchet and Rigollet [2011]

Let $\mathcal{X} \subset \mathbb{R}^d$, $\mathcal{A} = [n]$, (x_t) i.i.d. from some μ absolutely continuous w.r.t. Lebesgue. The reward for playing arm a under context x is drawn from some distribution $\nu_a(x)$ on $[0, 1]$ with mean function $f_a(x)$ which is assumed to be β-Holder smooth. Let $\Delta(x)$ be the "gap" function.

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One can achieve a regret of order $T\left(\frac{n \log(n)}{T}\right)^{\frac{\beta(\alpha+1)}{2\beta+d}}$, which is optimal at least in the dependency on T. It can be achieved by running Successive Elimination on an adaptively refined partition of the space, see Perchet and Rigollet [2011] for the details.

The online multi-class classification perspective after Kakade, Shalev-Shwartz, and Tewari [2008]

Here the loss is assumed to be of the following very simple form: $\ell_t(a) = \mathbb{1}\{a \neq a_t^*\}$. In other words using the context x_t one has to predict the best action (which can be interpreted as a *class*) $a_t^* \in [n]$.

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KSST08 introduces the *banditron*, a bandit version of the multi-class perceptron for this problem. While with full information the online multi-class perceptron can be shown to satisfy a "regret" bound on of order $\sqrt{T}$, the banditron attains only a regret of order $T^{2/3}$. See also Chapter 4 in Bubeck and Cesa-Bianchi [2012] for more on this.

Summary of advanced results

1. The optimal regret for the linear bandit problem is $\widetilde{O}(n\sqrt{T})$. In the Bayesian context Thompson Sampling achieves this bound. In the i.i.d. case one can use an algorithm based on the optimism in face of uncertainty together with concentration properties of the least squares estimator.
2. The i.i.d. algorithm can easily be modified to be computationally efficient, or to deal with sparsity in the unknown vector θ.
3. Extensions/variants: semi-bandit model, non-linear bandit (Lipschitz, Gaussian process, convex).
4. Contextual bandit is still a very active subfield of bandit theory.
5. Many important things were omitted. Example: knapsack bandit, see Badanidiyuru, Kleinberg and Slivkins [2013].

Some open problems we discussed

1. Prove the lower bound $\mathbb{E}R_T = \Omega(\sqrt{Tn\log(n)})$ for the adversarial n-armed bandit with adaptive adversary.
2. Guha and Munagala [2014] conjecture: for product priors, TS is a 2-approximation to the optimal Bayesian strategy for the objective of minimizing the number of pulls on suboptimal arms.
3. Find a "simple" strategy achieving the Bubeck and Slivkins [2012] best of both worlds result.
4. For the combinatorial bandit problem, find a strategy with regret at most $n^{3/2}\sqrt{T}$ (current best is $n^2\sqrt{T}$).
5. Is there a computationally efficient strategy for i.i.d. linear bandit with optimal $n\sqrt{T}$ gap-free regret and with $\log(T)$ gap-based regret?
6. Is there a natural framework to think about "bandits on top of bandits" (while keeping $\sqrt{T}$-regret)?